On Integrability of Some Quantum Control Systems

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Let $\mathcal{H}$ be an n-dimensional Hilbert space, $H : \mathcal{H} \to \mathcal{H}$ a time-dependent Hamiltonian and

$$i\hbar \frac{d}{dt} \left|\varphi_t\right\rangle = H(t) \left|\varphi_t\right\rangle \tag{1}$$

the Schrödinger equation. Assume that the Hamiltonian is given by

$$H(t) = H_0 + \sum_{i=1}^{\kappa} u_i(t) H_i$$

where u_i are time-dependent integrable scalar functions and H_i time-independent hermitian matrices.

If we fix the initial condition $\left|\varphi_0\right\rangle \in \mathcal{H}$, then the equation (1) has the unique solution, but its closed form is not known in general. In this presentation we give closed forms of solutions of (1) in some important cases. Our idea is to simultaneously decompose the matrices $H_0, H_1, ..., H_\kappa$ into blocks of low dimensions and solve (1) on the corresponding common invariant subspaces. We apply our method to give another proof of the existence of the Morris-Shore transformation [2]. We use the following crucial theorem from [1].

Denote by $\mathcal{M}_n(\mathbb{C})$ the algebra of $n \times n$ complex matrices.

Theorem 1. *Let $\pi(x_1, \ldots, x_m)$ be a polynomial in m variables $x_1, \ldots, x_m$ and p a fixed natural number. Assume that the identity $\pi = 0$ holds on the algebra $\mathcal{M}_p(\mathbb{C})$ but not on $\mathcal{M}_{p+1}(\mathbb{C})$. If the semi-simple algebra $\mathcal{A}(H_0, \ldots, H_\kappa)$ generated by $H_0, \ldots, H_\kappa$ satisfies $\pi = 0$, then the matrices $H_0, \ldots, H_\kappa$ can be simultaneously reduced to the block-diagonal form by a unitary matrix, and the size of each block is not greater than p.*

In the presentation we focus on the following examples studied in [1].

Example 1. *Assume that*

$$H(t) = u_1(t) \begin{pmatrix} I_1 & O \\ O & O_2 \end{pmatrix} + u_2(t) \begin{pmatrix} O_1 & V \\ V^* & O_2 \end{pmatrix} + u_3(t) \begin{pmatrix} O_1 & O \\ O & I_2 \end{pmatrix},$$

where O_k and I_k ($k = 1,2$) are the $n_k \times n_k$ zero matrix and the identity matrix, respectively, and V is a time-independent $n_1 \times n_2$ matrix with $n_1 + n_2 = n$. Then we can show that this system satisfies the conditions of Theorem 1 with $p = 2$, so we can block-diagonalise $H(t)$ by a unitary matrix into 1- or 2-dimensional blocks. The transformation from $H(t)$ into the block-diagonal form is called the Morris-Shore transformation, and our result gives another proof for the existence of this transformation.

Example 2. *A circulant matrix A is a matrix of the form*

$$A = \begin{pmatrix} a_0 & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & & \vdots \\ \vdots & & \ddots & a_{n-1} \\ a_{n-1} & \cdots & a_1 & a_0 \end{pmatrix},$$

where $a_i \in \mathbb{C}$. A circulant matrix can be diagonalised by a unitary matrix U which is independent of A, so if $H_0, \ldots, H_\kappa$ are all circulant matrices, then these matrices can be simultaneously diagonalised by U. Since we know the closed form of the solution of a 1-dimensional first order linear differential equation, we are able to give the closed form of the solution of original equation (1).

Example 3. *A Brownian matrix $A = (a_{ij})_{i,j}$ is a square matrix such that*

$$a_{i,j+1} = a_{i,j} \ (j > i), \ a_{i+1,j} = a_{i,j} \ (i > j) \ (\forall i, j).$$

A Brownian matrix can be block-diagonalised into one 2×2 block and $n-2$ number of 1-dimensional blocks by a unitary matrix U which is independent of A. Thus, if $H_0, \ldots, H_\kappa$ are all Brownian matrices, then we can give the closed form of the solution of equation (1).

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