Limit distributions for quantum walks on a line with Wigner rotation matrices

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There is an increasing interest in studying quantum walks lately. It is given by the potential application of quantum walks models in many fields regarding physics, information technology and chemistry. Quantum walks started as a theoretical concept describing processes in nature. Subsequently the experimental realizations using trapped atoms [1] and later cold ions [2, 3] and photons [4, 5] were conducted.

In the present paper we investigate a limit distribution of certain quantum walks. The work is motivated by paper by Miyazaki et al. [6], where authors introduced quantum walks models based on so called Wigner coins. Such coin depends on three Euler angles α, β, γ and the construction of these matrices comes from the quantum mechanical rotation operator and its irreducible matrix representation using Wigner formula [10]. Wigner walks are very interesting, since they exhibits trapping feature around the origin for odd dimensions. The trapping is connected to the existence of the point spectrum of the unitary propagator. This effect was studied for example for the Grover walk on the line. In [7], simple modifications of the Grover walks preserving localization are provided. One of these modifications for the three-state walk were also studied with respect to the limit distribution [8]. The results were further evolved and simplified in [9].

In [6], Wigner walks are analysed from the viewpoint of the limiting probability distribution. The authors derived general formulas for calculation of the limit distribution of quantum walkers and showed the exact results for several dimensions. It was shown that the velocity density does not depend on the angle α. We show that the dependence on γ can be eliminated through the simple rotation of the standard coin basis. Further, the total velocity density can be greatly simplified by a proper choice of the coin state basis, which we call optimal. The optimal basis also reveals some interesting features that are otherwise hidden.

As an example, we look at the two-state walk model where the walker can move one step to the right or left at each discrete moment. Instead of the angle β we use parameter ρ which express the maximal velocity of spreading of the walk. The relations between β and ρ are $\cos\frac{\beta}{2} = \rho, \sin\frac{\beta}{2} = \sqrt{1-\rho^2}$. The coin operator then reads

$$C = \begin{pmatrix} \rho & -\sqrt{1-\rho^2} \\ \sqrt{1-\rho^2} & \rho \end{pmatrix}.$$

In the standard basis, the velocity density gains form

$$\nu(v) = \frac{\sqrt{1-\rho^2}}{\pi(1-v^2)\sqrt{(\rho^2-v^2)}} \mathbf{I}_{\{|v|\leq|\rho|\}}(1+\mathcal{M}_1 v),$$

where $\mathbf{I}_{\{\dots\}}$ means identity on the given interval and

$$\mathcal{M}_1 = -|q_R|^2 + |q_L|^2 + +2\frac{\sqrt{1-\rho^2}}{\rho}\Re(q_R\overline{q}_L).$$

The coefficients $q_{R,L}$ arise from the expression of the initial coin state in the standard coin basis

$$|\psi_C\rangle = q_R|R\rangle + q_L|L\rangle, \quad |q_R|^2 + |q_L|^2 = 1.$$

The optimal basis changes $\mathcal{M}_1$ into

$$\mathcal{M}_1 = \frac{1}{\rho}(1-2|h_2|^2).$$

We see that except the simplification, the density now depends only on the probability $|h_2|^2$ to be in the optimal state.

The optimal basis is associated with elimination of the peaks in the probability distribution and is given by a specific linear combination of eigenvectors of the coin. The similar analysis can be done for higher-dimensional quantum walks. Nevertheless, there exist some differences between even and odd dimensions. In both cases, the optimal basis brings simplification of the velocity density expression. Moreover, one can immediately see some interesting situation that might occur for certain choices of the initial state.

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