Density matrices of scaled Husimi functions and some of their characteristic features

Vladimir Andreev,[1] Milena Davidović,[2] Ljubica Davidović,[3] Miloš Davidović,[4] and Dragomir Davidović[4,*]

[1] *N. Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia*
[2] *Faculty of Civil Engineering, University of Belgrade, Serbia*
[3] *Institute of Physics, University of Belgrade, Serbia*
[4] *Institute for Nuclear Sciences Vinča, University of Belgrade, Serbia*

Husimi function [1] $Q(q,p)$ of the density matrix $\hat{\rho}$ is defined as

$$Q(q,p) = \langle\alpha|\hat{\rho}|\alpha\rangle$$

where $|\alpha\rangle$ are coherent states of the linear harmonic oscilator. A characteristic and important property of the Husimi functions is the following. From all functions belonging to the class of Cohen functions [2] (one of the best known members of this class is the Wigner function [3]) only Husimi functions remain in their own class after scaling transformation $(q,p) \to (\lambda q, \lambda p)$ where $|\lambda| < 1$, and renormalization [4]. This means that $\lambda^2 Q(\lambda q, \lambda p)$ also describes some quantum state ie. a density matrix. In the present work we give the general formula for obtaining the density matrix from corresponding scaled Husimi function. We name the so obtained states the stretched states. We give this formula in two different representations which correspond to two representations of coherent states, namely: a coordinate representation of coherent states and the representation which uses Fock states.

We will now give some details about the streched Fock states. The Husimi function of the pure Fock state $|N\rangle$ is given by:

$$Q_N(q,p) = \langle\alpha|N\rangle\langle N|\alpha\rangle = \frac{1}{N!}|\alpha|^{2N} e^{-|\alpha|^2}. \quad (1)$$

Applying to the Husimi function Q_N the scaling transformation we get

$$Q_N^\lambda(q,p) = \lambda^2 Q_N(\lambda q, \lambda p) = \frac{\lambda^2}{N!}\lambda^{2N}|\alpha|^{2N} e^{-\lambda^2|\alpha|^2}. \quad (2)$$

Applying to the formula (2) our method for obtaining the density matrix from the scaled Husimi function, we obtain the stretched Fock state in the form:

$$\hat{\rho}_N^\lambda = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\frac{(N+k)!}{k!}(1-\lambda^2)^k|N+k\rangle\langle N+k|, \quad (3)$$

$$\lambda^2 < 1.$$

These Fock stretched states consist of pure states $|N+k\rangle$, $k = 0,1,2,\ldots\infty$. Each of these pure states $|N+k\rangle$ is present in the mixed state with the probability

$$c_k^N = \frac{\lambda^{2N+2}(N+k)!}{N!k!}(1-\lambda^2)^k. \quad (4)$$

The distribution of pure states is described by the negative binomial distribution [5]

$$f(k,r,p) = \begin{pmatrix} r+k-1 \\ k \end{pmatrix} p^r q^k;\ p+q=1;\ k=0,1,2,.. (5)$$

Using properties of this distribution, it is possible to find average values of quantities of interest such as the average photon number in a stretched state as

$$\langle n\rangle = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}(N+k)\frac{(N+k)!}{k!}(1-\lambda^2)^k = \frac{N+1}{\lambda^2} - 1,$$

and the dispersion of photon number as

$$\sigma_n = \langle n^2\rangle - (\langle n\rangle)^2 = \frac{(N+1)(1-\lambda^2)}{\lambda^4}.$$

We derived the Schrodinger-Robertson uncertainty relation for stretched states which reads

$$\sigma_{qq\lambda}\sigma_{pp\lambda} - \sigma_{qp\lambda}^2 = \frac{1}{\lambda^4}(\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 + \quad (6)$$

$$+\frac{1}{2}(1-\lambda^2)(\sigma_{qq}+\sigma_{pp}) + \frac{1}{4}(1-\lambda^2)^2) \geq \frac{1}{4\lambda^4}\hbar^2.$$

One can interpret the above inequality as if the scaling transform $(q,p) \to (\lambda q, \lambda p)$ provides an "effective Planck constant" value $\hbar_{eff} = \hbar/\lambda^2$. For small $\lambda^2 << 1$ the effective Planck constant satisfies the inequality

$$\hbar_{eff} >> \hbar. \quad (7)$$

The value of Planck constant $\hbar$ is responsible for purely quantum phenomenon such as quantum tunneling [6]. The well known quasiclassical formula for the transmission probability through the potential barrier $U(x)$ reads

$$D \approx \exp\left(-\frac{2}{\hbar}\int_a^b \sqrt{2m(U(x)-E)}dx\right). \quad (8)$$

Here m is the mass of particle and E is its energy.

It can be seen from (8) that for larger constant $\hbar_{eff}$ the quantum tunneling effect is more pronounced. Therefore, it can be concluded that stretched states are good candidates for possible realization of such an effect.

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* davidd@vinca.rs, corresponding author

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