The Garrison–Wong phase operator and the rotational covariance of phase states

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Every mode of a quantized boson (photon) field is described as a quantum harmonic oscillator. In this description one meets the quantum phase problem, which has been reviewed in the paper [1]. Using the photon annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, respectively, which obey the boson commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{1}$, with $\hat{1}$ an identity operator, the number operator is expressed as $\hat{n} = \hat{a}^\dagger \hat{a}$. Its eigenstates are the number states $|n\rangle$, $\hat{n}|n\rangle = n|n\rangle$. These states are rotationally invariant, $\exp(i\Delta\varphi\hat{n})|n\rangle = \exp(i\Delta\varphi n)|n\rangle$, with $\Delta\varphi$ a phase shift. The usual phase states $|\varphi\rangle$, $|\varphi\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty} e^{in\varphi}|n\rangle$, are not orthogonal. They are "rotationally covariant", $\exp(i\Delta\varphi)|\varphi\rangle = |\varphi + \Delta\varphi\rangle$. As they yield an identity resolution, $\hat{1} = \int_{\theta_0}^{\theta_0+2\pi} |\varphi\rangle\langle\varphi| d\varphi$, with θ_0 a reference phase, they enable one to "quantize" any function of the phase $M(\varphi)$, i.e., to introduce an operator $\hat{M} = \int_{\theta_0}^{\theta_0+2\pi} M(\varphi)|\varphi\rangle\langle\varphi| d\varphi$, where, e.g., $M(\varphi) = \cos\varphi, \sin\varphi, \varphi$. The Susskind-Glogower cosine and sine operators are mentioned in the paper [1] and the Garrison-Wong (GW) phase operator $\hat{\varphi}_{\theta_0}$ (notation by [2]) has been introduced in the paper [3] for $\theta_0 = -\pi$, in general it is studied in the paper [4]. For the determination of eigenstates of these operators corresponding to a real $M(\varphi)$, the general result of the mathematical paper [5] can be utilized. For the connection with the literature, we introduce the eigenstates of the GW phase operator, $|\theta\rangle_{\rm GW}$, $\hat{\varphi}_{-\pi}|\theta\rangle_{\rm GW} = \theta|\theta\rangle_{\rm GW}$ with the property ${}_{\rm GW}\langle\theta|\theta'\rangle_{\rm GW} = \delta(\theta - \theta')$, $\theta, \theta' \in (-\pi, \pi)$, where

$$|\theta\rangle_{\rm GW} = \sum_{n=0}^{\infty} \left[\frac{1}{\pi}\sin\left(\frac{\theta+\pi}{2}\right)\right]^{\frac{1}{2}} \phi_n(\theta)|n\rangle,$$

the functions $\phi_n(\theta)$ having been defined in the paper [3]. A proof of the fact that the eigenstates of the GW phase operator are not "rotationally covariant" can be realized as the study of the probability densities $P_{\rm GW}(\theta) \equiv |{}_{\rm GW}\langle\theta|n\rangle|^2$, $n = 0, 1, 2, 3$, [3, 4].

We treat the usual phase properties of the states $|\theta\rangle_{\rm GW}$. As these states are normalized to the Dirac δ function, in the study of the phase properties, we restrict ourselves to the first $s + 1$ terms of the expansion in the number-state basis and we normalize the truncated state. We calculate the phase characteristics, the preferred phase and the dispersion. These characteristics differ from θ and zero, respectively, for small s, but which is interesting, with increasing s, the preferred phase seems to approach θ and the dispersion appears to go to zero. The phase distributions seem to be "rotationally covariant". The numerical results do not exhibit a deviation of the GW phase state from the usual phase state under the assumption that we can restrict ourselves to the usual phase properties.

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