

# Single-Qudit Quantum Correlations as a Resource for Information Technologies

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The characteristics of quantum correlations in multipartite system states reflected by the existence of entropic and information inequalities like the subadditivity and strong subadditivity conditions are extended to the system without subsystems like a single qudit. In the states of a single qudit with an arbitrary spin  $j = (N - 1)/2$ , where the integer  $N = nm$ ,  $N = n_1 n_2 n_3$ , etc., with integer factors  $n, m, n_1 n_2, n_3$  associated with the density matrices  $\rho_{ss'} \equiv \rho_{s(jk),s'(j'k')} \equiv \rho_{s(pq),s'(p'q'l')} \equiv \dots$ . Here, the indices  $ss' = 1, 2, \dots, N$ ,  $jj' = 1, 2, \dots, n$ ,  $kk' = 1, 2, \dots, m$ ,  $pp' = 1, 2, \dots, n_1$ ,  $qq' = 1, 2, \dots, n_2$ ,  $ll' = 1, 2, \dots, n_3$ , etc., i.e., the matrix elements of the density matrix are labeled in the forms appropriate for bipartite, tripartite, etc., density matrices. Employing these multipartite-like representations of the density matrix of the single-qudit state, we can rewrite the inequalities and equalities known for the multipartite-system states for the single-qudit state.

As an example, we present the Araki–Lieb inequality for entropy of the three-level atomic state in the form

$$\begin{aligned} & -\text{Tr} \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} \\ & \geq \left| \text{Tr} \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{pmatrix} \right. \\ & \left. - \text{Tr} \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \right|; \end{aligned}$$

this is a new inequality for the qutrit state.

The subadditivity condition for the three-level atomic state reads

$$\begin{aligned} & -\text{Tr} \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} \\ & \leq -\text{Tr} \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{pmatrix} \\ & - \text{Tr} \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \ln \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}. \end{aligned}$$

The inequalities can be checked in the experiments where the density matrices of the qutrit states are measured.

For a single qudit with  $j = 3/2$ , we can write the entropic inequality analogous to the subadditivity condition known for a composite qudit–qutrit system by considering the density  $6 \times 6$ -matrix  $\tilde{\rho} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$ , where  $\rho$  is the density  $4 \times 4$ -matrix of the four-level atom, and we added appropriate extra zero matrix elements in order to obtain the  $6 \times 6$

matrix  $\tilde{\rho}$ . An analogous new subadditivity condition can be obtained for the two-qubit state in the form of entropic inequality for the qubit–qutrit state; this inequality can also be checked experimentally.

The presented new relations reflect the quantum correlations of the degrees of freedom of noncomposite systems. Since the correlations are analogous to the quantum correlations of the multipartite qubit systems, they can provide the resource for quantum technologies [1–3].

Quantum states are determined by quantum tomograms, which for qudit states are fair probability distributions  $w(m, u) = \langle m | u \rho u^\dagger | m \rangle$  of random spin projections  $m$  depending on the unitary matrix  $u$ . The inequality for two quantum tomograms  $w_1(m, u)$  and  $w_2(m, u)$  can be written [4, 5] in the form of the positivity condition for relative entropy:  $\sum_{m=-j}^j w_1(m, u) \ln \frac{w_1(m, u)}{w_2(m, u)} \geq 0$  known for classical probability distributions, but this inequality is written for quantum systems. The inequality is valid for an arbitrary unitary  $N \times N$ -matrix  $u$ .

Applying the positive map given by nonlinear channel to the density matrix  $\rho$ , i.e.,  $\rho \rightarrow \rho^p / (\text{Tr} \rho^p)$ , where  $p$  is a real number, one can get a new inequality for the purity parameters of three matrices:

$$(\text{Tr} \rho^p)^2 + (\text{Tr} \rho^{2p}) \geq (\text{Tr}_1 \rho^p)^2 + (\text{Tr}_2 \rho^p)^2.$$

If the matrix  $\rho$  is considered as the density matrix of the bipartite-system state  $\rho = \rho(1, 2)$ , the notation  $\text{Tr}_1$  and  $\text{Tr}_2$  means the partial tracing procedure. For the single qudit density  $N \times N$ -matrix  $\rho$  with  $N = nm$ , the notation corresponds to a formal partial tracing. At  $p = 1$ , we arrive at the inequality for the purity parameters  $1 + \mu_{12} \geq \mu_1 + \mu_2$ .

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