Thresholds for entanglement criteria in quantum information theory

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We approach the problem of separability of random bipartite quantum states obtained by tracing out one subsystem from a random, uniformly distributed, tripartite pure quantum state. This problem can be solved by applying on the second subsystem the *reduction map* $R : M_k(\mathbb{C}) \to M_k(\mathbb{C})$, $R(X) := I_k \cdot \mathrm{Tr}[X] - X$, and the corresponding separability test is called *reduction* (RED) *criterion* [5, 7]. Althrought it is weaker than PPT, the use of reduction criterion is justify by its connection with entanglement distillation: any state which violates the reduction criterion is distillable and, under certain protocols, it holds also conversely[8]. The separability problem was also approached by studying the class of *absolutely separable states* (ASEP), i.e. states that remain separable under any global unitary transformation [10], that means to find conditions on the spectrum that characterize absolutely separable states (constrains on the eigenvalues of a state ρ guaranteeing that ρ is separable with respect to any decomposition of the corresponding product tensor space [9]). We approach the problem of separability and absolute-separability from a different perspective. We aim to derive thresholds for the reduction and absolute reduction criteria and to give a complete picture of threshold points for the class of entanglement criteria. The threshold point is defined in the following sense: given a random mixed state $\rho_{AB} \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$, obtained by partial tracing over $\mathbb{C}^s$ a uniformly distributed, pure quantum state $x \in \mathbb{C}^n \otimes \mathbb{C}^k \otimes \mathbb{C}^s$, where the s-dimensional space is treated like an inaccessible environment, we ask for the probability that the state satisfies an entanglement criterion. The threshold phenomenon was introduced by Aubrun to study the PPT criterion [2]. Our main contribution is to compute thresholds for the dimension of the system being trace out, so that the resulting bipartite quantum state satisfies the reduction criterion in different asymptotic regimes. Our results completes similar results obtained in[11]. We consider as well the basis-independent version of the reduction criterion (the absolute-reduction criterion), computing thresholds for the corresponding eigenvalue sets, using techniques from random matrix theory. Finally, we gather and compare the known values for thresholds corresponding to different entanglement criteria.

	Balanced regime $n,k \to \infty$	Unbalanced regime $m = \min(n,k)$ fixed, $\max(n,k) \to \infty$	
SEP	$n^3 \lesssim s \lesssim n^3 \log^2 n$ [1, n=k]	$mnk \lesssim s \lesssim mnk \log^2(nk)$ [1]	
PPT	$s \sim cnk$	$s \sim cnk$	
	$c = 4$ [2, $n = k$]	$c = 2 + 2\sqrt{1 - \frac{1}{m^2}}$ [4]	
RLN	$s \sim cnk$	s fixed	
	$c = (8/3\pi)^2$ [3, $n = k$]	$s = m^2$ [3]	
RED	$s \sim cn$	$m = n$, s fixed	$m = k$, $s = cnk$
	$c = 1$	$s = n$	$c = \frac{(1+\sqrt{k+1})^2}{k(k-1)}$ [11]

TABLE I. Thresholds for separability vs. entanglement

	Balanced regime $n,k \to \infty$	Unbalanced regime $m = \min(n,k)$ fixed, $\max(n,k) \to \infty$	
APPT	$s \sim c \min(n,k)^2 nk$	$s \sim cnk$	
	$c = 4$ [6]	$c = \left(m + \sqrt{m^2 - 1}\right)^2$ [6]	
ARED	$s \sim cnk$	For $m = n$, $s \sim ck$	For $m = k$, $s = cnk$
	$c = 1$	$c = n - 2$	$c = \left(1 + \frac{2}{k} + \frac{2}{k}\sqrt{k+1}\right)^2$

TABLE II. Thresholds for related sets

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