# Maximally unpolarized, pure, quantum states

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In classical optics an unpolarized state is a state for which the Stokes vector $(\langle\hat{S}_1\rangle, \langle\hat{S}_2\rangle, \langle\hat{S}_3\rangle)$ vanishes. Such a state, if described in a quantum optics picture, is a maximally mixed state. (Hence, it is maximally mixed in any basis.) However, it has been known for some time that there exist pure states with a vanishing Stokes vector. This phenomenon have been denoted "hidden polarization" [1]. The word "hidden" alludes to the fact that although the state's Stokes vector vanishes, if the Stokes operator *variance* is measured in different directions on the Poincaré sphere the variance is not isotropic. E.g., for the hidden polarization state $|1,1\rangle$, i.e., a two-photon state with one photon each in any two orthogonal polarization modes, the Stokes operator variance has unit visibility as a function of the appropriate polarization transformation.

One can then ask if there are pure states that also have an isotropic variance, that is, they are unpolarized to second order? A natural extension of this question is to ask if there exist states for which $\langle\hat{S}_{\mathbf{n}}^k\rangle$, where $\hat{S}_{\mathbf{n}} \equiv \mathbf{n}\cdot(\hat{S}_1, \hat{S}_2, \hat{S}_3)$ and $\mathbf{n}$ is a unit vector on the Poincaré sphere, is isotropic (independent of the direction $\mathbf{n}$ on the Poincaré sphere) for $k = 1, 2, \ldots, K$? If so, such a state can be said to be $K$th order unpolarized.

We answer both questions in the affirmative and conjecture that it is possible to find pure states unpolarized to any order $K$. A natural subsequent question is if there is any relation between the state's photon number $N$ and to what degree $K$ such a state can be unpolarized? Indeed there is such a connection, but the relation is rather irregular and is closely connected to the seemingly unrelated geometrical problem of distributing $N$ points on a sphere as "symmetrically" as possible. For example, for $N = 48$ one can find states unpolarized from 1st to 9th order, but we conjecture that no 48-photon state exists that is unpolarized to the 10th order.

An interesting geometrical connection exist between maximally unpolarized states and so-called spherical t-designs [2]. The latter is a configuration of $N$ points on a sphere, such that the average value of any $t$th order polynomial on the sphere is given by the average value of the polynomial at the $N$ points. The $t$-designs that maximize $t$ for a given $N$ are of particular interest in this context. What we find is that in every case we have examined, if there exist an $N$ point spherical $t$-design, one can find an $N$ photon state unpolarized to order $K = t$ but none to $K > t$. It is interesting to note that such states' Majorana representation, which maps any $N$-photon, pure, polarization state to a configuration of $N$ points on a sphere [3], does in general not correspond to the spherical $t$-design configuration except when $N$ is small so that there exist limited degrees of freedom to find the most symmetric configuration. However, in spite of the non-perfect correspondence, the Majorana representation provides a powerful tool to find maximally unpolarized states.

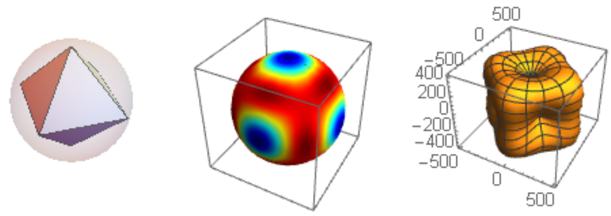

FIG. 1. Left: The Majorana representation of the $N = 6$ maximally unpolarized state $(|5,1\rangle + |1,5\rangle)/\sqrt{2}$ is given by the vertices of a regular octahedron inscribed in the Poincaré sphere. Middle: The SU(2) Q-function representation of the state in pseudocolor (blue denote the minima, red the maxima). The Q-function's minima correspond to the Majorana points. The state is unpolarized to the third order. Right: The first non-isotropic moment $\langle\hat{S}_{\mathbf{n}}^4\rangle$.

The SU(2) coherent states are the most classical states in a polarization sense. They have the largest polarization dipole, octopole, etc., contribution for a given photon number. This is also suggested by their number-state expansion. All $N$-photon SU(2) coherent states are simply polarization rotations of the state $|N,0\rangle$, that is, e.g., a state with all its excitation in a specific polarization, e.g., linear, vertical polarization. (A $\lambda/4$-plate will make all $N$ photons circularly polarized, which is another SU(2) coherent state.) Since the maximally unpolarized states have vanishing polarization dipole, quadropole, etc. contributions they can be said to be the most non-classical polarization states. Not surprisingly, they are also highly bipartite entangled. As for applications, they seem to be suitable for sensitive detection of arbitrary polarization rotations.

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