Information-Entropic Inequalities for Density Matrices and Possibilities of Their Verification

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The states of qudits are associated with the density $N\times N$-matrix ρ for the both composite and noncomposite systems. The $N\times N$-matrices ρ^p with $N = nm$ and $p \neq 0$ can be presented in the block form: $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, where the blocks a_{jk} are $m\times m$-matrices. Introducing the matrices $\rho_1 = \begin{pmatrix} \mathrm{Tr}\, a_{11} & \mathrm{Tr}\, a_{12} & \cdots & \mathrm{Tr}\, a_{1n} \\ \mathrm{Tr}\, a_{21} & \mathrm{Tr}\, a_{22} & \cdots & \mathrm{Tr}\, a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{Tr}\, a_{n1} & \mathrm{Tr}\, a_{n2} & \cdots & \mathrm{Tr}\, a_{nn} \end{pmatrix}$ and $\rho_2 = \sum_{k=1}^{n} a_{kk}$, one can obtain the following inequality:

$$\big(\mathrm{Tr}\,\rho^p\big)^q + \mathrm{Tr}\,(\rho^{pq}) \geq \mathrm{Tr}\,(\rho_1)^q + \mathrm{Tr}\,(\rho_2)^q, \qquad q > 1. \quad (1)$$

If the matrix ρ is the density matrix of a bipartite-system state $\rho(1,2)$, then $\rho_1 = \mathrm{Tr}_2\big(\rho(1,2)\big)^p$, $\rho_2 = \mathrm{Tr}_1\big(\rho(1,2)\big)^p$, and inequality (1) is similar to the Minkowski-type inequalities studied in [1, 2] in view of the approach elaborated in [3] and developed in [4]. Inequalities (1) are valid for systems of qudits, as well as for a single qudit.

It is worth noting that inequality (1) holds also for $N\times N$-matrices ρ, where $N \neq nm$. In this case, we construct $N'\times N'$-matrix $\rho' = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$ with extra zero matrix elements, such that $N' = nm$. Then one repeats the block construction of matrix ρ' and obtains the inequality for new matrices ρ_1' and ρ_2'.

In the case of multimode states described by the Gaussian Wigner function corresponding to the state density matrix ρ with nonnegative eigenvalues λ_n, the real purity parameter of the state $\mathrm{tr}\,\rho^p = \mu(q) = \sum_n \lambda_n^q$ for $q \geq 1$, satisfies the inequality $\mu(q) \leq 1$. If the quadrature variances and covariances determining the Wigner function violate the uncertainty relations, the inequality $\mu(q) \leq 1$ is violated. For example, $\mathrm{Tr}\,\rho^2$, being dependent on variances and covariances, is larger than unity for the "state" with the Gaussian function, $W(x,p) = 2\,a \exp(-ax^2 - ap^2)$ if $a > 1$. For Gaussian states of classical systems with the same probability density in the phase space, the violation of the inequality is permitted, since the uncertainty relation does not exist in the classical domain.

The tomogram $w(m,u)$ of the state ρ of the system of qudits is determined as a set of diagonal matrix elements of the matrix $\langle m \mid u\rho u^\dagger \mid m\rangle$, where u is the unitary $N\times N$-matrix. If u_0 is the unitary matrix with columns associated with the eigenvectors of the matrix ρ corresponding to eigenvalues r_k, $k = 1, 2, \ldots, N$, the tomographic-probability distribution $w(m,u)$ has the form of the probability N-vector $\vec{w}(u) = |uu_0|^2 \vec{r}$, where the vector $\vec{r}$ has components r_k and the matrix elements of the matrix $|a|^2$ are defined as $|a|^2_{jk} = |a_{jk}|^2$. The tomogram determines the matrix ρ. For an arbitrary quantum state, the tomographic probability distribution satisfies the inequality

$$0 \leq \sum_{k=1}^{N} w_k(u) \ln w_k(u) - \sum_{s=1}^{n} \Omega_s^{(1)}(u) \ln \Omega_s^{(1)}(u) - \sum_{l=1}^{m} \Omega_l^{(2)}(u) \ln \Omega_l^{(2)}(u),$$

where the probability vectors $\vec{\Omega}^{(1)}(u)$ and $\vec{\Omega}^{(2)}(u)$ are obtained from the probability vector $\vec{w}(u)$ by the portrait method [5]. The above inequality is the subadditivity condition, which means the nonnegativity of the mutual information. These inequalities can be verified in experiments with superconducting circuits [6, 7].

For the case of a single qudit with $j = 3/2$, one can obtain an analog of the inequalities known for two qubits like the subadditivity condition. For a single qudit, one can obtain an analog of the strong subadditivity condition, which also can be verified using a superconducting circuit.

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Abstract

New inequalities for the tomographic mutual information valid for composite and noncomposite systems are found for the tomographic-probability distributions determining the density matrices of the system states. The inequality for an arbitrary density matrix similar to the Minkowski-type inequality is obtained. The inequality is valid for an arbitrary dimension of the qudit system. The possibility to check the inequalities in in the experiments with superconducting circuits are discussed.

Keywords: density matrix, information, superconducting circuit, entropic inequality.