Variational approach to the ground state of n-level atoms interacting with ℓ modes of an electromagnetic field

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The ground state of n-level atoms interacting with ℓ modes of a radiation field is constructed by means of a variational procedure. As test function, we select the tensorial product of coherent states associated to the electromagnetic field and the totally symmetric representation of U(n) for the atomic part.

By means of this trial state we calculate the expectation value of the Hamiltonian per atom, which is called the energy surface. This is a function of the complex variables α_{jk} and γ_k, which define the coherent states plus the parameters of the Hamiltonian. These control parameters are the frequencies of the radiation modes Ω_{jk}, the frequencies of the atomic levels ω_k, and the dipolar strengths μ_{jk} between the levels k and j. We have taken $\mu_{jk} = \mu_{kj}$, and the condition on the labels of the photonic modes, $j < k$, with $k = 2, \cdots, n$.

Minimising the energy surface one determines the existence of two regimes for the ground state of the system:

- The normal region, where the ground state is determined by all the atoms in the lower atomic energy level ($\omega_1 = 0$) without photons (vacuum state); implying an energy value $\mathcal{E}_N = 0$.
- The collective part, where there are excited atoms and photons following multinomial and Poisson distribution functions, respectively.

These regimes determine the quantum phase diagram of the system. There are sudden changes in the structure of the ground state when the borders are crossed.

For n-level atomic configurations, one has $\frac{n(n-1)}{2}$ different energy surfaces associated to the superradiant regime, that is,

$$\mathcal{E}_{1k} = -\frac{(-4\mu_{1k}^2 + \Omega_{1k}\omega_k)^2}{16\mu_{1k}^2\Omega_{1k}},$$
$$\mathcal{E}_{jk} = \omega_j - \frac{(-4\mu_{jk}^2 + \Omega_{jk}(\omega_k - \omega_j))^2}{16\mu_{1k}^2\Omega_{1k}},$$

where the first expression exists for the dipolar strengths satisfying $4\mu_{1k}^2 \geq \omega_k\Omega_{1k}$, and the second for $4\mu_{jk}^2 \geq (\omega_k - \omega_j)\Omega_{jk}$, with $2 \leq j < k$. However some of these energy surfaces are not available due to the selection rules of the dipolar matrix elements between atomic levels. Additionally, one can identify these conditions with the quantum phase transitions ocurring in 2-level atomic configurations interacting with one mode of the radiation field, that is the Dicke model. Thus the minimum energy surface is given by

$$E_{min} = \min\left\{\mathcal{E}_N, \{\mathcal{E}_{1k}\}, \{\mathcal{E}_{jk}\}\right\}. \quad (1)$$

From the expressions for the minimum energy value in the different regions, and following the Ehrenfest classification, one finds the order of transitions in the phase diagram as

$$E_N \leftrightarrows E_{1k}\,,\text{Second},\quad E_N \leftrightarrows E_{jk}\,,\text{First},\quad E_{jk} \leftrightarrows E_{j'k'}\,,\text{First}.$$

It is straightforward to see that one may reduce the system to an equivalent one with $n-1$ levels, where the atomic variables ϱ_j are replaced by new variables η_j which, upon finding their critical points, lead to an equivalent algebraic system of $n-2$ levels and the process may be continued until we reach a two-level system with one radiation mode whose properties, including the complete structure of the phase diagram, have been studied extensively [2].

Therefore one can conclude that the quantum phase diagram for n-level atoms interacting with ℓ radiation modes is constituted by the normal and superradiant regimes, but the last one is divided into monochromatic parts where only two atomic levels are participating together with one mode of electromagnetic radiation. A first order transition indicates that at least one physical quantity (expectation value of the number of photons, atomic populations, etc.) has a discontinuity at the border of the phase diagram, which is obtained through the expressions $\mathcal{E}_N = \mathcal{E}_{1k}$, $\mathcal{E}_N = \mathcal{E}_{jk}$, and $\mathcal{E}_{1k} = \mathcal{E}_{jk}$. where again some of these equalities cannot be established for the existence of forbidden transitions.

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